Exercise 75

Find the parabola with equation $y = ax^2 + bx$ whose tangent line at $(1, 1)$ has equation $y = 3x - 2$.

Solution

The tangent line at $(1, 1)$ has a slope of 3. Take the derivative of the parabola equation.

$$y' = \frac{d}{dx}(ax^2 + bx)$$

$$= \frac{d}{dx}(ax^2) + \frac{d}{dx}(bx)$$

$$= a\frac{d}{dx}(x^2) + b\frac{d}{dx}(x)$$

$$= a(2x) + b(1)$$

$$= 2ax + b$$

Use the fact that at $x = 1$, the slope of the parabola is 3.

$$y'(1) = 2a + b = 3 \tag{1}$$

Also, use the fact that at $x = 1$, $y = 1$.

$$y(1) = a(1)^2 + b(1) = a + b = 1 \tag{2}$$

Solve equations (1) and (2) for a and b.

$$a = 2 \qquad b = -1$$

Therefore, the parabola with equation $y = ax^2 + bx$ whose tangent line at $(1, 1)$ has equation $y = 3x - 2$ is

$$y = 2x^2 - x.$$